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A fourth-order compact difference scheme on face centered cubic grids with multigrid method for solving 2D convection diffusion equation

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Abstract

We present a fourth-order compact finite difference scheme on the face centered cubic (FCC) grids for the numerical solution of the two-dimensional convection diffusion equation. The seven-point formula is defined on a regular hexagon, where the strategy of directional derivative is employed to make the derivation procedure straightforward, efficient, and concise. A corresponding multigrid method is developed to solve the resulting sparse linear system. Numerical experiments are conducted to verify the fourth-order convergence rate of the derived discretization scheme and to show that the fourth-order compact difference scheme is computationally more efficient than the standard second-order central difference scheme.

MSC: 65F10; 65N06; 65N22; 65N55; 76D07

Keywords: Convection diffusion equation; Multigrid method; Face centered cubic grid; Fourth-order compact scheme

1. Introduction

Many transport phenomena and physical processes, including fluid flow, heat and mass transfer, can be modeled by a general convection diffusion equation, which describes the convection diffusion characteristics of various physical quantities, such as momentum, energy, concentrations, etc. In this paper,

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we concern ourselves with the numerical solution of two-dimensional convection diffusion equation of the form

$$u_{xx} + u_{yy} + P(x, y)u_x + Q(x, y)u_y = F(x, y), \tag{1}$$

defined on a convex domain Ω with suitable boundary conditions. We assume that the solution $u(x, y)$, the convection coefficients $P(x, y)$, $Q(x, y)$, and the forcing function $F(x, y)$ are sufficiently smooth on Ω.

It is known that Eq. (1) is important for both computational fluid dynamics (CFD) and diffusive heat transfer in heterogeneous systems. For CFD, the magnitude of the convection coefficients determines the ratio of the convection to diffusion, and is referred to as the Reynolds number in fluid mechanics, denoted by *Re*. For large *Re*, Eq. (1) is said to be convection dominated, otherwise it is diffusion dominated. For heat transfer, the convection coefficients are related to the ratio of the gradient of thermal conductivity to the thermal conductivity.

Among the initial steps of numerical modeling of physical problems is to choose a suitable computational grid to discretize the governing partial differential equations. In the current case, we are concerned with discretizing the 2D convection diffusion equation on face centered cubic (FCC) grids. Boghosian [1] used the FCC grid for lattice gas modeling. Xu [2] used the FCC grid for the numerical solution of partial differential equations because it allows efficient, stable, and robust implementation of the multidimensional finite differencing method (MDFD), which is a general methodology for the numerical solution of partial differential equations defined on irregular domains. The grid works particularly well for MDFD because it allows for efficient multivariate interpolation. Xu refers to the FCC grid as a hypersphere-close-pack (HCP) grid because the location of the computational nodes corresponds to the location of the centers of densely packed equidiameter hyperspheres (at least in 2D, 3D, and 4D), as depicted in Fig. 1 for the case of 2D. In two-dimensions, the FCC grid is the equilateral triangular mesh, as shown in Fig. 2. A similar 2D FCC grid was discussed by Kantorovich and Krylov [3] in 1958. The FCC grids are highly structured. One of the easiest ways [2] to construct an FCC grid is to color the nodes of a Cartesian grid with two colors so that no two neighbors in the directions of the coordinate axes have the same colors, and then remove all of the nodes of one of the colors. The remaining nodes form the FCC grid. In 2D, the FCC grid has a seven-point finite difference stencil. For uniform grid spacing, the outer points are equidistant from the center and form a hexagon around the center (see Fig. 3).

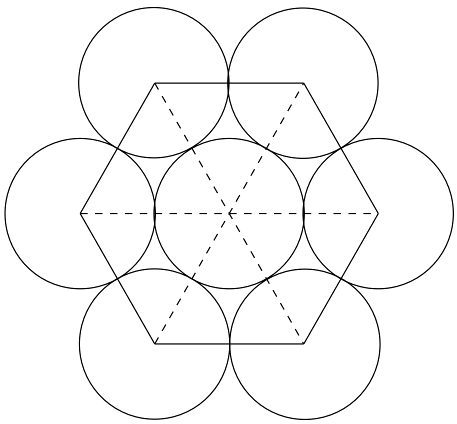

Fig. 1. The 2D hypersphere-close-packing (close-packed circles) [2].

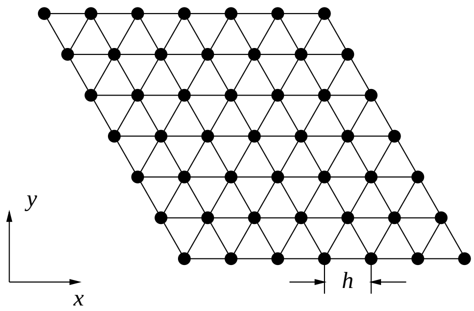

Fig. 2. The 2D FCC grid (the equilateral triangular mesh) with mesh size h.

One important issue to be considered in our study is how to choose an appropriate finite difference scheme for discretizing the 2D convection diffusion equation on the FCC grid to achieve good numerical stability and high accuracy approximations. One of the approaches to obtaining accurate solutions is to use higher-order or locally exact discretization methods for the convection diffusion equation [5,6]. These higher-order schemes have good numerical stability and yield approximations of high accuracy. Gupta et al. [8] derived a fourth-order compact finite difference approximation for Eq. (1). They also extended their derivations to general two-dimensional elliptic equations in [9] on standard Cartesian grid. The idea behind their methods is to utilize the Taylor series expansions of all the functions involved in Eq. (1). Several other authors using alternative techniques have put forward a number of similar fourth-order compact finite difference schemes for the 2D convection diffusion equations [4,7,10]. All these formulas may seem to be in different forms, but there is a clear indication that they are mathematically equivalent. These discretization schemes have been shown by analytical results and numerical experiments to have better numerical stability and higher accuracy approximations than the standard second-order finite difference schemes [11–15].

Another important issue in numerical computation of partial differential equations is to efficiently solve the sparse linear systems arising from the discretized convection diffusion equations. For large scale and complicated problems, direct solution methods based on Gaussian elimination are expensive and not

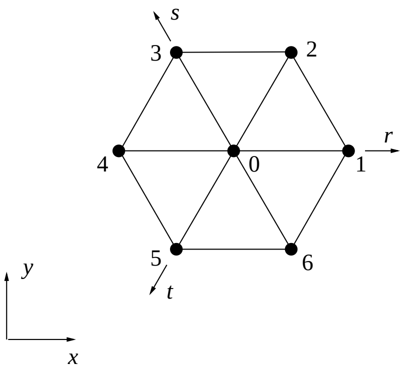

Fig. 3. The seven-point finite difference stencil in the 2D FCC grid.

efficient in terms of memory usage and CPU time. Thus iterative methods are generally believed to be the more viable means in such situations. The studies by Zhang [14,16] indicate that the fourth-order compact schemes work well with fast iterative solution methods, e.g. the multigrid methods. Such fourth-order compact difference schemes with the multigrid solution methods have been found computationally more efficient than the traditional second-order central difference scheme for the diffusion dominated problems and for the problems where the convection is not very strong [17]. To obtain computed solution with a given accuracy, the fourth-order compact schemes using relatively coarser discretization may be hundreds of times faster and may use less memory than the standard second-order central difference scheme.

Focusing on the 2D Poisson equation, Kantorovich and Krylov [3] derived fourth-order approximations using what they called the method of nets based on the regular triangular and hexagonal nets as well as on the square net. Their regular triangular net is the same as the 2D FCC grid which we described earlier. The main idea of their scheme is to apply the Taylor's formula to the 2D Poisson equation on the seven-point stencil in their regular triangular net. Since the grid greatly benefits MDFD with domain embedding [2], we intend to extend the idea into the discretization of the 2D convection diffusion equation on the FCC grid but to use a totally different derivation scheme. Our derivation procedure appears to be clearer and more efficient. Furthermore, we will use the fourth-order compact difference scheme to develop a multigrid method involving the FCC grid to efficiently solve the resulting sparse linear system.

This paper is organized as follows. In Section 2, we present a fourth order compact finite difference scheme on the FCC grid for the 2D convection diffusion equation. Section 3 contains the solution strategies for the resulting linear system, including the multigrid method and the multigrid intergrid transfer operators. The results of numerical experiments are shown in Section 4 to verify the accuracy of the fourth-order compact difference scheme and the computational efficiency of the multigrid solution method. Section 5 gives brief concluding remarks.

2. Fourth-order compact scheme

In this section, we illustrate the derivation of the fourth-order compact finite difference approximations on the FCC grid. This scheme, based on Taylor series expansions, is similar to the one presented by Kantorovich and Krylov [3] for the Poisson equation, but we utilize the concept of directional derivative in the derivation procedure.

The discretization is assumed to be performed on the 2D uniform FCC grid with mesh size h, which is the length of the side of the equilateral triangles. The finite difference formula for a mesh point (x, y) identified by the number 0 involves its six nearest nodal points, which are denoted by the numbers $1, 2, \ldots, 6$, respectively, as shown in Fig. 3. As we described earlier, the outer six points are equidistant from the center and form a hexagon around the center. The unknown nodal values of u at the seven grid points are denoted by $u_0, u_1, \ldots, u_6$. We also assume that the nodal values of P, Q, and F are known.

Thanks to the characteristics of the hexagon, we define three unit vectors to denote the three different directions shown in Fig. 3,

$$\vec{r} = \vec{x}, \quad \vec{s} = -\frac{1}{2}\vec{x} + \frac{\sqrt{3}}{2}\vec{y}, \quad \vec{t} = -\frac{1}{2}\vec{x} - \frac{\sqrt{3}}{2}\vec{y}. \tag{2}$$

Since the directional derivatives of u in the direction $\vec{r}$, $\vec{s}$, and $\vec{t}$ are given by, respectively,

$$u_r = \vec{\mathrm{grad}}(u) \cdot \vec{r} = u_x,$$

$$u_s = \vec{\mathrm{grad}}(u) \cdot \vec{s} = -\frac{1}{2}u_x + \frac{\sqrt{3}}{2}u_y,$$

and

$$u_t = \vec{\mathrm{grad}}(u) \cdot \vec{t} = -\frac{1}{2}u_x - \frac{\sqrt{3}}{2}u_y,$$

we have

$$u_r + u_s + u_t = 0. \tag{3}$$

Similarly, we can also get

$$u_{rr} = \vec{\mathrm{grad}}(u_r) \cdot \vec{r} = u_{xx},$$

$$u_{ss} = \vec{\mathrm{grad}}(u_s) \cdot \vec{s} = \frac{1}{4}u_{xx} - \frac{\sqrt{3}}{2}u_{xy} + \frac{3}{4}u_{yy},$$

and

$$u_{tt} = \vec{\mathrm{grad}}(u_t) \cdot \vec{t} = \frac{1}{4}u_{xx} + \frac{\sqrt{3}}{2}u_{xy} + \frac{3}{4}u_{yy}.$$

Thus, we have

$$u_{rr} + u_{ss} + u_{tt} = \tfrac{3}{2}(u_{xx} + u_{yy}).$$

We note that $u_x = u_r = -(u_s + u_t)$ from Eq. (3) and $u_s - u_t = \sqrt{3}u_y$. Eq. (1) can be rewritten as

$$u_{rr} + u_{ss} + u_{tt} + pu_s + qu_t = f, \tag{4}$$

where

$$p = \frac{\sqrt{3}}{2}Q - \frac{3}{2}P, \quad q = -\frac{\sqrt{3}}{2}Q - \frac{3}{2}P, \quad f = \frac{3}{2}F.$$

Eq. (4) is the one for which we will derive the fourth-order compact finite difference scheme.

Using Taylor series, it can be shown that the following relations hold.

$$\begin{aligned}
&\delta_r u = u_r + \tfrac{1}{6}u_{rrr}h^2 + O(h^4), \quad \delta_{rr} u = u_{rr} + \tfrac{1}{12}u_{rrrr}h^2 + O(h^4),\\
&\delta_s u = u_s + \tfrac{1}{6}u_{sss}h^2 + O(h^4), \quad \delta_{ss} u = u_{ss} + \tfrac{1}{12}u_{ssss}h^2 + O(h^4),\\
&\delta_t u = u_t + \tfrac{1}{6}u_{ttt}h^2 + O(h^4), \quad \delta_{tt} u = u_{tt} + \tfrac{1}{12}u_{tttt}h^2 + O(h^4),
\end{aligned} \tag{5}$$

where we use the notations

$$\delta_r u = \frac{u_1 - u_4}{2h}, \quad \delta_{rr} u = \frac{u_1 + u_4 - 2u_0}{h^2},$$
$$\delta_s u = \frac{u_3 - u_6}{2h}, \quad \delta_{ss} u = \frac{u_3 + u_6 - 2u_0}{h^2},$$
$$\delta_t u = \frac{u_5 - u_2}{2h}, \quad \delta_{tt} u = \frac{u_2 + u_5 - 2u_0}{h^2}.$$

In order to obtain a fourth-order compact difference scheme for Eq. (4), only second-order approximations are needed to apply to the third directional derivatives u_{rrr}, u_{sss}, u_{ttt}, and the sum of $u_{rrrr} + u_{ssss} + u_{tttt}$ in (5). This is due to the fact that all of the terms mentioned have an h^2 factor. To achieve this goal, we first derive the second-order approximation of the directional derivative u_{rst}.

From (5), we can obtain

$$\delta_r u + \delta_s u + \delta_t u = u_r + u_s + u_t + \tfrac{1}{6}h^2(u_{rrr} + u_{sss} + u_{ttt}) + \mathrm{O}(h^4). \tag{6}$$

After repeatedly taking directional derivatives on Eq. (3), we can get

$$u_{rrr} + u_{srr} + u_{trr} = 0, \qquad u_{rss} + u_{sss} + u_{tss} = 0, \qquad u_{rtt} + u_{stt} + u_{ttt} = 0, \tag{7}$$

$$u_{rst} = -(u_{srr} + u_{rss}), \qquad u_{rst} = -(u_{rtt} + u_{trr}), \qquad u_{rst} = -(u_{tss} + u_{stt}), \tag{8}$$

and

$$u_{rrr} + u_{sss} + u_{ttt} = -(u_{rrs} + u_{ssr} + u_{ttr} + u_{rrt} + u_{sst} + u_{tts}) = 3u_{rst}. \tag{9}$$

Substituting Eq. (9) back into Eq. (6), we have

$$u_{rst} = \frac{2\delta_r u + 2\delta_s u + 2\delta_t u}{h^2} + \mathrm{O}(h^2), \tag{10}$$

which is a second-order approximation. Similarly, after a few manipulations of directional derivative on Eq. (3) and using Eq. (4), we can get

$$\begin{aligned} u_{rr} &= u_{st} - \tfrac{1}{2}(f - pu_s - qu_t), \\ u_{ss} &= u_{rt} - \tfrac{1}{2}(f - pu_s - qu_t), \\ u_{tt} &= u_{rs} - \tfrac{1}{2}(f - pu_s - qu_t). \end{aligned} \tag{11}$$

Then we take directional derivatives on both sides of the three equations in (11) in the directions $\vec{r}$, $\vec{s}$, and $\vec{t}$, respectively. We obtain

$$\begin{aligned} u_{rrr} &= u_{rst} - \tfrac{1}{2}[f_r - (pu_s)_r - (qu_t)_r], \quad & u_{rrrr} &= u_{rrst} - \tfrac{1}{2}[f_{rr} - (pu_s)_{rr} - (qu_t)_{rr}], \\ u_{sss} &= u_{rst} - \tfrac{1}{2}[f_s - (pu_s)_s - (qu_t)_s], \quad & u_{ssss} &= u_{rsst} - \tfrac{1}{2}[f_{ss} - (pu_s)_{ss} - (qu_t)_{ss}], \\ u_{ttt} &= u_{rst} - \tfrac{1}{2}[f_t - (pu_s)_t - (qu_t)_t], \quad & u_{tttt} &= u_{rstt} - \tfrac{1}{2}[f_{tt} - (pu_s)_{tt} - (qu_t)_{tt}]. \end{aligned} \tag{12}$$

The above derived representation (12) and Eq. (10) as well as (11) indicate that the directional derivatives u_{rrr}, u_{sss}, and u_{ttt} have second-order approximations. Using Eq. (3), we can get $u_{rrst} + u_{rsst} + u_{rstt} = 0$.

Thus the sum,

$$u_{rrrr}+u_{ssss}+u_{tttt}=-\frac{1}{2}[f_{rr}+f_{ss}+f_{tt}-(pu_s)_{rr}-(pu_s)_{ss}-(pu_s)_{tt}-(qu_t)_{rr}-(qu_t)_{ss}-(qu_t)_{tt}], \tag{13}$$

also has second-order approximation after we do some manipulations on the right hand side of the expression, since all the third directional derivatives can be appropriately handled to reach second-order approximation using (3), and (7)–(10).

If we replace (5) with the representations (12) and (13), the discretization of Eq. (4) can be represented as

$$\begin{aligned}&(1-C-G)u_1+\left(1-\tfrac{1}{2}qh+\tfrac{1}{2}Bh-E+G\right)u_2+\left(1+\tfrac{1}{2}ph-\tfrac{1}{2}Ah-D-G\right)u_3\\&\quad+(1-C+G)u_4+\left(1+\tfrac{1}{2}qh-\tfrac{1}{2}Bh-E-G\right)u_5+\left(1-\tfrac{1}{2}ph+\tfrac{1}{2}Ah-D+G\right)u_6\\&\quad-(6-2C-2D-2E)u_0=\tfrac{1}{24}h^2(f_1+f_2+f_3+f_4+f_5+f_6+18f_0)+H,\end{aligned}$$

where

$$\begin{aligned}&A=\tfrac{1}{24}h^2[(pq)_t-p^2q-2qp_t-(\delta_{rr}p+\delta_{ss}p+\delta_{tt}p)],\\&B=\tfrac{1}{24}h^2[(pq)_s-q^2p-2pq_s-(\delta_{rr}q+\delta_{ss}q+\delta_{tt}q)],\quad C=-\tfrac{1}{12}h^2(pq+p_t+q_s),\\&D=-\tfrac{1}{12}h^2(\tfrac{1}{2}p^2+p_s+q_r);\quad E=-\tfrac{1}{12}h^2(\tfrac{1}{2}q^2+p_r+q_t);\quad G=\tfrac{1}{6}h(p+q);\\&H=\tfrac{1}{24}h^4(pqf+2pf_s+2qf_t-(pf)_s-(qf)_t).\end{aligned}$$

The derivation procedure indicates that the obtained seven-point compact difference scheme on the FCC grid has a fourth-order truncation error. It is important to point out that, in the actual computation, the derivatives of the functions p, q, and f can be approximated using the standard second-order central difference scheme. Only the nodal values of the functions will be needed.

3. Multigrid method

The finite difference equations at all interior grid points form a large sparse linear system, which needs to be solved efficiently. The solution cost of this sparse linear system usually dominates the total cost of solving the discretized partial differential equations.

One advantage of the fourth-order compact difference scheme is that it has the same number of stencil points as that of the second-order central difference scheme. Thus, except for some extra preprocessing cost to precompute the stencil coefficients, which is only done once, the coefficinet matrix of the discretized convection diffusion equation, derived from the fourth-order compact difference scheme, has the same nonzero structure as that from the second-order central difference scheme. It follows that we should expect similar cost for solving the linear systems arising from discretized convection diffusion equation using the two discretization schemes, if a direct solution method is used. If an iterative method is used, the cost of per iteration should be the same, regardless of which discretization scheme is used. The one that converges faster will be less expensive in terms of the total solution time for solving the sparse linear system.

Direct solution methods based on Gaussian elimination are robust, but they may have a huge memory demand which makes the computation expensive and is not always feasible. We resort to iterative methods to solve the sparse linear systems.

Among the many iterative techniques, the multigrid method offers convergence rate that is independent of the grid size and is very effective for solving large scale discretized elliptic problems [18,19]. Various multigrid methods and convergence acceleration techniques have been designed to solve the 2D convection diffusion equations discretized by the fourth-order compact difference schemes on the Cartesian grid [11,16]. In this section, we present a multigrid V-cycle algorithm for solving the resulting discretized convection diffusion equation on the FCC grid.

We denote Ω_h as the original and the finest FCC grid with mesh size h and use the notations u_h, f_h, L_h, I_{2h}^h, and I_h^{2h} to denote the approximate solution, the right hand side vector, the finite difference operator, the interpolation operator, and the restriction operator, on the Ω_h grid, respectively. We give our multigrid V-cycle procedure for the FCC grid system as follows:

1. Perform one or more relaxation(s) on Ω_h using the fourth-order compact difference operator on
$$L_h u_h = f_h.$$
2. Compute and project the residual from Ω_h to Ω_{2h} using the standard full-weighting residual restriction operator
$$I_h^{2h} = \frac{1}{12}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$
3. Repeat steps 1 and 2 until $h = 1/2$ and then solve the system on $\Omega_{1/2}$ directly.
4. Add correction from Ω_{2h} to the approximate solution on Ω_h by interpolation
$$u_h := u_h + I_{2h}^h u_{2h},$$
where the interpolation operator is chosen to be $I_{2h}^h = (I_h^{2h})^{\mathrm{T}}$.
5. Perform one or more relaxation(s) on Ω_h using the fourth-order compact difference operator on
$$L_h u_h = f_h.$$
6. Repeat steps 4 and 5 until h returns to the original fine grid size.

The multigrid V-cycle algorithm terminates when the 2-norm of the residual vector is reduced by a certain order of magnitude.

4. Numerical results

We compare our fourth-order compact difference scheme with the second order central difference scheme for solving two test problems. The first test problem is chosen as a Poisson equation. The second one is chosen as a convection diffusion equation with variable coefficients P and Q. A standard multigrid method is applied for solving the discretized problems. We use one Gauss–Seidel relaxation for each of the pre- and post-smoothing operators. The initial guess is the zero vector. In our numerical experiments, we set the stopping criterion as $||r_k||_2/||r_0||_2 < 10^{-10}$, where r_k is the residual vector of the approximate solution after the kth iterations. The error reported is the maximum absolute errors over the entire discrete grid. The experiments are performed on a Sun Ultra 5 workstation. The code is written in Fortran 77 programming

Table 1
Computational results for Test Problem 1

h	Fourth-order scheme				Second-order scheme			
	Error	Order	Iteration	cpu	Error	Order	Iteration	cpu
1/8	2.0×10^{-3}		10	0.00	7.7×10^{-2}		10	0.00
1/16	1.2×10^{-4}	4.05	11	0.02	1.9×10^{-2}	2.01	11	0.02
1/32	7.7×10^{-6}	3.96	13	0.14	4.7×10^{-3}	2.02	13	0.14
1/64	4.8×10^{-7}	4.00	13	0.66	1.2×10^{-3}	1.96	13	0.65
1/128	3.0×10^{-8}	4.00	13	2.94	3.0×10^{-4}	2.00	13	2.93
1/256	1.9×10^{-9}	3.98	13	13.27	7.4×10^{-5}	2.02	13	13.19

language and is run in double precision. In the following numerical experiments, we consider the domain that is described as in Fig. 2, which can be defined as $\Omega = \{(x, y)|0 \le y \le 1, -y/2 \le x \le 1 - (y/2)\}$.

For Test Problem 1, we set the convection coefficients $P = Q = 0$ on Ω in Eq. (1). The exact solution is chosen as $u = \cos(4x + 6y)$. The right hand side function F and the Dirichlet boundary condition are prescribed according to the exact solution. The test results are tabulated in Table 1.

It is clear from Table 1 that the fourth-order compact difference scheme computes more accurate solution than the second-order central difference scheme does. Multigrid method with both schemes exhibits grid independent convergence rate. It is also clear that, with the same grid size h, the fourth-order compact difference scheme is only slightly more expensive to compute with. For Test Problem 1, the convection coefficients are identically zero, the preprocessing cost for computing the stencil coefficients is low. Multigrid method with the fourth-order compact difference scheme and the second-order central difference scheme has the same convergence rate and the cost of each iteration is essentially the same for both schemes, per our previous discussion. The advantage of the fourth-order compact difference scheme is overwhelming, as the solution computed from the fourth-order compact difference scheme is much more accurate.

For Test Problem 2, we choose in Eq. (1) the convection coefficients as $P = 10(2y - 1)(1 - x^2)$ and $Q = -10y(y - 1)(1 - 2x)$. The exact solution is chosen as $u = \exp(-10(x - (1/2))^2 - y^2)$. The right hand side function F and the Dirichlet boundary condition are prescribed according to the exact solution. The computational data are listed in Table 2.

Again, we see that the computed solution from the fourth-order compact difference scheme exhibits the fourth-order accuracy rate, while that from the second-order central difference scheme has a second-order

Table 2
Computational results for Test Problem 2

h	Fourth-order scheme				Second-order scheme			
	Error	Order	Iteration	cpu	Error	Order	Iteration	cpu
1/8	5.5×10^{-4}		10	0.01	1.6×10^{-2}		11	0.01
1/16	3.4×10^{-5}	4.01	12	0.06	4.0×10^{-3}	2.00	12	0.03
1/32	2.2×10^{-6}	3.95	13	0.22	1.0×10^{-3}	2.00	13	0.14
1/64	1.4×10^{-7}	3.97	13	0.93	2.5×10^{-4}	2.00	13	0.67
1/128	8.7×10^{-9}	4.01	13	3.95	6.3×10^{-5}	1.98	13	2.95
1/256	5.4×10^{-10}	4.01	13	17.14	1.5×10^{-5}	2.07	13	13.29

accuracy rate. The convergence rate of the multigrid method does not seem to be affected by the nonzero convection coefficients.

For Test Problem 2, the preprocessing cost increases a little bit, as the non constant stencil coefficients need to be computed. The stencil coefficients of the fourth-order compact difference scheme are more complicated than those of the second-order central difference scheme. As a result, the implementation of the fourth-order compact difference is somehow more expensive. However, this increased cost is totally offset by the increased accuracy of the computed solution. The fourth-order compact difference scheme can compute more accurate solution faster on a coarser grid. This fact is evident, according to the data in Table 2.

5. Concluding remarks

We derived a fourth-order compact finite difference scheme on a face centered cubic (FCC) grid for the 2D convection diffusion equation. In order to efficiently make use of the characteristics of the 2D FCC grid, the strategy of directional derivative is employed in the derivation procedure of the fourth-order compact discretization scheme. The derivation procedure appears to be more concise, more straightforward, and easier to implement than simply applying the Taylor series expansions to the convection diffusion equation.

A multigrid algorithm with the fourth-order compact difference scheme is developed and is used successfully to solve the 2D convection diffusion problems on the FCC grid. Numerical experiments show that the proposed multigrid method works well on our test problems. The advantage of the fourth-order compact difference scheme on the FCC grid is that it uses only seven stencil points in the approximation scheme, and the same number of stencil points is also used in the lower-order (central difference) scheme. With a Cartesian grid in 2D, the second-order central difference scheme has a five-point stencil, while the fourth order compact difference scheme has a nine-point stencil [17].

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